Solution of HW 3

1.

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$$

$$\det(\lambda I - A) = \begin{pmatrix} \lambda & -2 \\ -2 & \lambda-3 \end{pmatrix} = (\lambda-4)(\lambda+1)$$

$\lambda_1 = 4 \quad \lambda_2 = -1$

$(\lambda_1 I - A)\vec{v_1} = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{v_1} = (1,2)$$

$(\lambda_2 I - A)\vec{v_2} = 0$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{v_2} = (-2,1)$$

Let $P = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

Then $P^{-1}AP = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$\det(\lambda I - B) = \det\begin{pmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{pmatrix} = (\lambda-4)(\lambda+2)$$

$\lambda_1 = 4 \quad \lambda_2 = -2$

$(\lambda_1 I - B)\vec{v_1} = 0 \qquad \vec{v_1} = (1,1)$

$(\lambda_2 I - B)\vec{v_2} = 0 \qquad \vec{v_2} = (1,-1)$

Let $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$P^{-1}BP = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

2 a. $z = x^2y + y^2$ $\quad F = x^2y + y^2 - z$

$V_x = (1, 0, 2xy)$ $\quad V_y = (0, 1, x^2+2y)$

$$\vec{N} = \frac{\nabla F}{|\nabla F|} = \frac{(2xy, x^2+2y, -1)}{\sqrt{4x^2y^2 + (x^2+2y)^2 + 1}}$$

$$\vec{N_x} = \frac{1}{(1+4x^2y^2+(x^2+2y)^2)^{\frac{3}{2}}}\big(-xy(8xy^2+4x(x^2+2y)),$$

$$2(x+2x^3y^2-4xy^3),\ 4xy^2+2x(x^2+2y)\big)$$

At the point $(0,0,0)$

$V_x = (1,0,0)$ $\quad V_y = (0,1,0)$

$\vec{N_x} = (0,0,0)$

$\vec{N_y} = (0,2,0)$

$$S_{(0,0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

b. $\det(\lambda I - S) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda-2 \end{pmatrix} = 0 \Rightarrow \lambda_1 = 0 \quad \lambda_2 = 2$

principal curvature $k_1 = 0$ $\quad k_2 = 2$

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$$f(x,y)=\sqrt{56x^2-8y^2-16x-31}+1-8x$$

$$\frac{\partial f}{\partial x}=-8+\frac{8(-1+7x)}{\sqrt{56x^2-8y^2-16x-31}}$$

$$\frac{\partial f}{\partial y}=\frac{-8y}{\sqrt{56x^2-8y^2-16\cancel{y}x-31}}$$

$$\begin{cases}\frac{\partial f}{\partial x}=0\\ \frac{\partial f}{\partial y}=0\end{cases}\Rightarrow\begin{cases}x=\frac{16}{7}\\ y=0\end{cases}$$

Critical point $\left(\frac{16}{7},0\right)$

$$f_{xx}=-\frac{8(225+56y^2)}{(-31-16x+56x^2-8y^2)^{\frac{3}{2}}}=\frac{-8}{15}$$

$$f_{xy}=\frac{4(-16+112x)y}{(-31-16x+56x^2-8y^2)^{\frac{3}{2}}}=0$$

$$f_{yy}=\frac{8(31+16x-56x^2)}{(-31-16x+56x^2-8y^2)^{\frac{3}{2}}}=-\frac{8}{15}$$

$$H=\begin{pmatrix}-\frac{8}{15} & 0\\ 0 & -\frac{8}{15}\end{pmatrix}$$

The point $\left(\frac{16}{7},0\right)$ is a ~~saddle~~ local maximum point.

24. $f(x,y)=e^{2x}\cos y$

$f_x=2e^{2x}\cos y \quad f_y=-e^{2x}\sin y$

$\nabla f=0 \Rightarrow \cancel{x}\begin{cases}\cos y=0\\ \sin y=0\end{cases}$ This is impossible.

There is no ~~cr~~ critical point, hence no local extreme or saddle point.

36 $f(x,y) = 48xy - 32x^3 - 24y^2$

$f_x = 48y - 96x^2$

$f_y = 48x - 48y$

$\nabla f = 0 \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$ or $\begin{cases} x=\frac{1}{2} \\ y=\frac{1}{2} \end{cases}$

Along the boundary $x=0$

$f(x,y) = -24y^2$

$f(0,0) = \boxed{0}$ $f(0,1) = \boxed{-24}$

Along $y=0$ $f(x,y) = -32x^3$

$f(1,0) = \boxed{-32}$

$x=1$ $f(x,y) = -24(y-1)^2 - 8$

$f(1,0) = \boxed{-32}$ $f(1,1) = \boxed{-8}$

$y=1$ $f(x,y) = 48x - 32x^3 - 24$

$\frac{\partial f(x,1)}{\partial x} = 48(-2x^2+1) = 0 \Rightarrow x = \frac{\sqrt{2}}{2}$

$f(0,1) = \boxed{-24}$ $f(1,1) = -8$ $f(\frac{\sqrt{2}}{2},1) = -40 + 24\sqrt{2}$
$= \boxed{-6.06}$

$f(\frac{1}{2},\frac{1}{2}) = \boxed{2}$

Global maximum $f(\frac{1}{2},\frac{1}{2}) = 2$
minimum $f(1,0) = -32$.

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$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=0 \end{cases}$$

$\Rightarrow (y,x)=\lambda(2x,2y)$

$x^2+y^2-10=0$

$(x,y)=(\sqrt{5},-\sqrt{5})$, $(\sqrt{5},\sqrt{5})$ $(-\sqrt{5},\sqrt{5})$ $(-\sqrt{5},-\sqrt{5})$

maximum: $f(\sqrt{5},\sqrt{5})=5$ $\quad$ minimum $f(\sqrt{5},-\sqrt{5})=-5$

$f(-\sqrt{5},-\sqrt{5})=5$ $\quad$ $f(-\sqrt{5},\sqrt{5})=-5$

3. $f(x,y)=49-x^2-y^2$

$g(x,y)=x+3y-10$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=0 \end{cases} \Rightarrow (x,y)=(1,3)$$

$f(1,3)=39$ $\quad$ maximum $f(1,3)=39$ no minmum

7. a. ~~f(x+y~~ $f(x,y)=x+y$ $\quad g(x,y)=xy-16$

$\nabla f=(1,1)$ $\quad \nabla g=(x,y)$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=0 \end{cases} \quad (x,y)=(4,4) \text{ or } (-4,-4)$$

since $x>0$ $y>0$ critical point is $(4,4)$

minimum $f(4,4)=8$

b. $f(x,y)=xy \quad g(x,y)=x+y-16$

$\nabla f=(y,x) \quad \nabla g=(1,1)$

$\left.\begin{array}{l}\nabla f=\lambda\nabla g \\ g(x,y)=0\end{array}\right. \Rightarrow (x,y)=\cancel{(4,4)}\ (8,8)$

~~mi~~ maximum $f(4,4)=\cancel{16}\ 64$